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## Drapeau Theorem for Differential Systems

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### 1. INTRODUCTION

This talk is concerned with the Drapeau theorem for differential systems. By a differential system  $(R, D)$ , we mean a distribution  $D$  on a manifold  $R$ , i.e.,  $D$  is a subbundle of the tangent bundle  $T(R)$ . The *derived system*  $\partial D$  of  $D$  is defined, in terms of sections, by

$$\partial D = D + [D, D].$$

where  $D = \Gamma(D)$  denotes the space of sections of  $D$ . In general  $\partial D$  is obtained as a subsheaf of the tangent sheaf of  $R$  (for the precise argument, see e.g. [Y1], [BCG3]). Moreover higher derived systems  $\partial^i D$  are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put  $\partial^0 D = D$  by convention. In this talk, a differential system  $(R, D)$  is called regular if  $\partial^i D$  are subbundles of  $T(M)$  for every  $i \geq 1$ .

We say that  $(R, D)$  is an  $m$ -flag of length  $k$ , if  $(R, D)$  is regular and has a derived length  $k$ , i.e.,  $\partial^k D = T(R)$ ;

$$D \subset \partial D \subset \cdots \subset \partial^{k-2} D \subset \partial^{k-1} D \subset \partial^k D = T(R),$$

such that  $\text{rank } D = m + 1$  and  $\text{rank } \partial^i D = \text{rank } \partial^{i-1} D + m$  for  $i = 1, \dots, k$ . In particular  $\dim R = (k + 1)m + 1$ .

Especially  $(R, D)$  is called a *Goursat flag* (un drapeau de Goursat) of length  $k$  when  $m = 1$ . Historically, by Engel, Goursat and Cartan, it is known that a Goursat flag  $(R, D)$  of length  $k$  is locally isomorphic, at a generic point, to the canonical system  $(J^k(M, 1), C^k)$  on the  $k$ -jet spaces of 1 independent and 1 dependent variable (for the definition of the canonical system  $(J^k(M, 1), C^k)$ , see §2). The characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [B] for the first order systems and in [Y1] and [Y2] for higher order systems for  $n$  independent and  $m$  dependent variables. However, it was first explicitly exhibited by A. Giaro, A. Kumpera and C. Ruiz in [GKR] that a Goursat flag of length 3 has singularities and the research of singularities of Goursat flags of length  $k$  ( $k \geq 3$ ) began as in [M1]. To this situation, R. Montgomery and M. Zhitomirskii constructed the “Monster Goursat manifold” by successive applications of the “Cartan prolongation of rank 2 distributions [BH]” to a surface and showed that every germ of a Goursat flag  $(R, D)$  of length  $k$  appears in this “Monster Goursat manifold” in [MZ]

, by first exhibiting the following Sandwich Lemma for  $(R, D)$ ;

$$\begin{array}{ccccccc} D & \subset & \partial D & \subset \cdots \subset & \partial^{k-2} D & \subset \partial^{k-1} D \subset & \partial^k D = T(R) \\ \cup & & \cup & & \cup & & \\ \text{Ch}(D) & \subset & \text{Ch}(\partial D) & \subset & \text{Ch}(\partial^2 D) & \subset \cdots \subset & \text{Ch}(\partial^{k-1} D) \end{array}$$

where  $\text{Ch}(\partial^i D)$  is the Cauchy characteristic system of  $\partial^i D$  and  $\text{Ch}(\partial^i D)$  is a subbundle of  $\partial^{i-1} D$  of corank 1 for  $i = 1, \dots, k-1$ . Here the *Cauchy Characteristic System*  $\text{Ch}(C)$  of a differential system  $(R, C)$  is defined by

$$\text{Ch}(C)(x) = \{X \in C(x) \mid X \rfloor dw_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \text{ for } i = 1, \dots, s\},$$

where  $C = \{\omega_1 = \dots = \omega_s = 0\}$  is defined locally by defining 1-forms  $\{\omega_1, \dots, \omega_s\}$ . Moreover, after [MZ], P. Mormul defined the notion of a *special  $m$ -flag* of length  $k$  for  $m \geq 2$  to characterize those  $m$ -flags which are obtained by successive applications of the “generalized Cartan prolongation” to the space of 1-jets of 1 independent and  $m$  dependent variables.

The main purpose of this talk is first to clarify the procedure of “Rank 1 Prolongation” of an arbitrary differential system  $(R, D)$  of rank  $m+1$ , and to give good criteria for an  $m$ -flag of length  $k$  to be special, i.e., to be locally isomorphic to the  $k$ -th Rank 1 Prolongation  $(P^k(M), C^k)$  of a manifold  $M$  of dimension  $m+1$ . More precisely we will show for an  $m$ -flag of length  $k$  for  $m \geq 3$ ; Corollary 5.8.

Moreover we will show for an  $m$ -flag of length  $k$  for  $m \geq 2$ , Corollary 6.3.

For this purpose, we will first review the geometric construction of jet spaces in §2 and clarify the procedure of Rank 1 Prolongation in §3. In §4, we will analyze the notion of a special  $m$ -flag of length  $k$  and reestablish the local characterization of  $(P^k(M), C^k)$  by utilizing the Realization Lemma [Y1]. In §5 and §6, we will introduce the main theorem (the Drapeau Theorem) for an  $m$ -flag of length  $k$ .

## 2. GEOMETRIC CONSTRUCTION OF JET SPACES

In this section, we will briefly recall the geometric construction of jet bundles in general, following [Y1] and [Y2], which is our basis for the later considerations.

Let  $M$  be a manifold of dimension  $m+n$ . Fixing the number  $n$ , we form the space of  $n$ -dimensional *contact elements* to  $M$ , i.e., the Grassmann bundle  $J(M, n)$  over  $M$  consisting of  $n$ -dimensional subspaces of tangent spaces to  $M$ . Namely,  $J(M, n)$  is defined by

$$J(M, n) = \bigcup_{x \in M} J_x, \quad J_x = \text{Gr}(T_x(M), n),$$

where  $\text{Gr}(T_x(M), n)$  denotes the Grassmann manifold of  $n$ -dimensional subspaces in  $T_x(M)$ . Let  $\pi : J(M, n) \rightarrow M$  be the bundle projection. The *canonical system*  $C$  on  $J(M, n)$  is, by definition, the differential system of codimension  $m$  on  $J(M, n)$  defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M, n)) \mid \pi_*(v) \in u\} \subset T_u(J(M, n)) \xrightarrow{\pi_*} T_x(M),$$

where  $\pi(u) = x$  for  $u \in J(M, n)$ .

Let us describe  $C$  in terms of a canonical coordinate system in  $J(M, n)$ . Let  $u_o \in J(M, n)$ . Let  $(x_1, \dots, x_n, z^1, \dots, z^m)$  be a coordinate system on a neighborhood  $U'$  of  $x_o = \pi(u_o)$  such that  $dx_1, \dots, dx_n$  are linearly independent when restricted to  $u_o \subset T_{x_o}(M)$ . We put  $U = \{ u \in \pi^{-1}(U') \mid dx_1|_u, \dots, dx_n|_u \text{ are linearly independent} \}$ . Then  $U$  is a neighborhood of  $u_o$  in  $J(M, n)$ . Here  $dz^\alpha|_u$  is a linear combination of  $dx_i|_u$ 's, i.e.,  $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$ . Thus, there exist unique functions  $p_i^\alpha$  on  $U$  such that  $C$  is defined on  $U$  by the following 1-forms;

$$\omega^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha dx_i \quad (\alpha = 1, \dots, m),$$

where we identify  $z^\alpha$  and  $x_i$  on  $U'$  with their lifts on  $U$ . The system of functions  $(x_i, z^\alpha, p_i^\alpha)$  ( $\alpha = 1, \dots, m, i = 1, \dots, n$ ) on  $U$  is called a *canonical coordinate system* of  $J(M, n)$  subordinate to  $(x_i, z^\alpha)$ .

$(J(M, n), C)$  is the (geometric) 1-jet space and especially, in case  $m = 1$ , is the so-called contact manifold. Let  $M, \hat{M}$  be manifolds of dimension  $m + n$  and  $\varphi : M \rightarrow \hat{M}$  be a diffeomorphism. Then  $\varphi$  induces the isomorphism  $\varphi_* : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$ , i.e., the differential map  $\varphi_* : J(M, n) \rightarrow J(\hat{M}, n)$  is a diffeomorphism sending  $C$  onto  $\hat{C}$ . The reason why the case  $m = 1$  is special is explained by the following theorem of Bäcklund.

**Theorem(Bäcklund)** *Let  $M$  and  $\hat{M}$  be manifolds of dimension  $m + n$ . Assume  $m \geq 2$ . Then, for an isomorphism  $\Phi : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$ , there exists a diffeomorphism  $\varphi : M \rightarrow \hat{M}$  such that  $\Phi = \varphi_*$ .*

The essential part of this theorem is to show that  $F = \text{Ker } \pi_*$  is the covariant system of  $(J(M, n), C)$  for  $m \geq 2$ . Namely an isomorphism  $\Phi$  sends  $F$  onto  $\hat{F} = \text{Ker } \hat{\pi}_*$  for  $m \geq 2$ . For the proof, we refer the reader to Theorem 1.4 in [Y2].

In case  $m = 1$ , it is a well known fact that the group of isomorphisms of  $(J(M, n), C)$ , i.e., the group of contact transformations, is larger than the group of diffeomorphisms of  $M$ . Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number  $m$  of dependent variables is 1 or greater.

(1) Case  $m = 1$ . We should start from a contact manifold  $(J, C)$  of dimension  $2n + 1$ , which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space  $(L(J), E)$  as follows: We consider the Lagrange-Grassmann bundle  $L(J)$  over  $J$  consisting of all  $n$ -dimensional integral elements of  $(J, C)$ ;

$$L(J) = \bigcup_{u \in J} L_u \subset J(J, n),$$

where  $L_u$  is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space  $(C(u), d\omega)$ . Here  $\omega$  is a local contact form on  $J$ . Namely,  $v \in J(J, n)$  is an integral element if and only if  $v \subset C(u)$  and  $d\omega|_v = 0$ ,

where  $u = \pi(v)$ . Let  $\pi : L(J) \rightarrow J$  be the projection. Then the canonical system  $E$  on  $L(J)$  is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where  $\pi(v) = u$  for  $v \in L(J)$ . We have  $\partial E = \pi_*^{-1}(C)$  and  $\text{Ch}(C) = \{0\}$  (cf. [Y1]). Hence we get  $\text{Ch}(\partial E) = \text{Ker } \pi_*$ , which implies the Bäcklund theorem for  $(L(J), E)$  (cf. [Y1]).

Now we put

$$(J^2(M, n), C^2) = (L(J(M, n)), E),$$

where  $M$  is a manifold of dimension  $n + 1$ .

(2) Case  $m \geq 2$ . Since  $F = \text{Ker } \pi_*$  is a covariant system of  $(J(M, n), C)$ , we define  $J^2(M, n) \subset J(J(M, n), n)$  by

$$J^2(M, n) = \{n\text{-dim. integral elements of } (J(M, n), C), \text{ transversal to } F\},$$

$C^2$  is defined as the restriction to  $J^2(M, n)$  of the canonical system on  $J(J(M, n), n)$ .

Now the higher order (geometric) jet spaces  $(J^{k+1}(M, n), C^{k+1})$  for  $k \geq 2$  are defined (simultaneously for all  $m$ ) by induction on  $k$ . Namely, for  $k \geq 2$ , we define  $J^{k+1}(M, n) \subset J(J^k(M, n), n)$  and  $C^{k+1}$  inductively as follows:

$$J^{k+1}(M, n) = \{n\text{-dim. integral elements of } (J^k(M, n), C^k), \text{ transversal to } \text{Ker } (\pi_{k-1}^k)_* \},$$

where  $\pi_{k-1}^k : J^k(M, n) \rightarrow J^{k-1}(M, n)$  is the projection. Here we have

$$\text{Ker } (\pi_{k-1}^k)_* = \text{Ch}(\partial C^k),$$

and  $C^{k+1}$  is defined as the restriction to  $J^{k+1}(M, n)$  of the canonical system on  $J(J^k(M, n), n)$ . Then we have ([Y1], [Y2])

$$\begin{array}{ccccccc} C^k & \subset \cdots \subset & \partial^{k-2} C^k & \subset \partial^{k-1} C^k \subset \partial^k C^k = T(J^k(M, n)) \\ \cup & & \cup & & \cup \end{array}$$

$$\{0\} = \text{Ch}(C^k) \subset \text{Ch}(\partial C^k) \subset \cdots \subset \text{Ch}(\partial^{k-1} C^k) \subset F$$

where  $\text{Ch}(\partial^{i+1} C^k)$  is a subbundle of  $\partial^i C^k$  of corank  $n$  for  $i = 0, \dots, k-2$  and, when  $m \geq 2$ ,  $F$  is a subbundle of  $\partial^{k-1} C^k$  of corank  $n$ . The transversality conditions are expressed as

$$C^k \cap F = \text{Ch}(\partial C^k) \quad \text{for } m \geq 2, \quad C^k \cap \text{Ch}(\partial^{k-1} C^k) = \text{Ch}(\partial C^k) \quad \text{for } m = 1$$

By the above diagram together with the rank condition, Jet spaces  $(J^k(M, n), C^k)$  can be characterized as higher order contact manifolds as in [Y1] and [Y2].

Here we observe that, if we drop the transversality condition in our definition of  $J^k(M, n)$  and collect all  $n$ -dimensional integral elements, we may have some singularities in  $J^k(M, n)$  in general. However, since every 2-form vanishes on 1-dimensional subspaces, in case  $n = 1$ , the integrability condition for  $v \in J(J^{k-1}(M, 1), 1)$  reduces to  $v \subset C^{k-1}(u)$  for  $u = \pi_{k-1}^k(v)$ . Hence, in this case, we can safely drop

the transversality condition in the above construction as in the next section, which constitutes the key construction for the Drapeau theorem in later considerations.

### 3. RANK 1 PROLONGATION

Let  $(R, D)$  be a differential system, i.e.,  $R$  is a manifold of dimension  $s + m + 1$  and  $D$  is a subbundle of  $T(R)$  of rank  $m + 1$ . Starting from  $(R, D)$ , we define  $(P(R), \hat{D})$  as follows (cf. [BH]):

$$P(R) = \bigcup_{x \in R} P_x \subset J(R, 1),$$

where

$$P_x = \{1\text{-dim. integral elements of } (R, D)\} = \{u \subset D(x) \mid 1\text{-dim. subspaces}\} \cong \mathbb{P}^m.$$

Let  $p : P(R) \rightarrow R$  be the projection. We define the canonical system  $\hat{D}$  on  $P(R)$  by

$$\hat{D}(u) = p_*^{-1}(u) = \{v \in T_u(P(R)) \mid p_*(v) \in u\} \subset T_u(P(R)) \xrightarrow{p_*} T_x(R),$$

where  $p(u) = x$  for  $u \in P(R)$ .

We call  $(P(R), \hat{D})$  the *prolongation of rank 1* (or *Rank 1 Prolongation* for short) of  $(R, D)$ . Then  $P(R)$  is a manifold of dimension  $2m + s + 1$  and  $\hat{D}$  is a differential system of rank  $m + 1$ . In case  $(R, D) = (M, T(M))$ , we have  $(P(M), \hat{D}) = (J(M, 1), C)$ . Moreover

$$J^2(M, 1) \subset P(J(M, 1)) \subset J(J(M, 1), 1)$$

As for the prolongation of rank 1, we have

**Proposition 3.1.** *Let  $(R, D)$  be a differential system of rank  $m + 1$  and let  $(P(R), \hat{D})$  be the prolongation of rank 1 of  $(R, D)$ . Then  $\hat{D}$  is of rank  $m + 1$ ,  $\partial \hat{D} = p_*^{-1}(D)$  and  $\text{Ch}(\hat{D})$  is trivial. Moreover, if  $\text{Ch}(D)$  is trivial, then  $\text{Ch}(\partial \hat{D})$  is a subbundle of  $\hat{D}$  of corank 1.*

This proposition implies that, starting from any differential system  $(R, D)$ , we can repeat the procedure of Rank 1 Prolongation. Let  $(P^1(R), D^1)$  be the prolongation of rank 1 of  $(R, D)$ . Then  $(P^k(R), D^k)$  is defined inductively as the prolongation of rank 1 of  $(P^{k-1}(R), D^{k-1})$ , which is called *k-th prolongation of rank 1* of  $(R, D)$ . Moreover, starting from a manifold  $M$  of dimension  $m + 1$ , we put

$$(P^k(M), C^k) = (P(P^{k-1}(M)), \hat{C}^{k-1})$$

where  $(P^1(M), C^1) = (J(M, 1), C)$ . When  $m = 1$ ,  $(P^k(M), C^k)$  are called “monster Goursat manifolds” in [MZ].

Here we observe that the above proposition also implies

**Proposition 3.2.** *Let  $(R, D)$  be an  $m$ -flag of length 1, i.e.,  $\dim R = 2m + 1$ ,  $\text{rank } D = m + 1$  and  $\partial D = T(R)$ . Then the  $k$ -th prolongation  $(P^k(R), D^k)$  of rank 1 of  $(R, D)$  is an  $m$ -flag of length  $k + 1$ . Namely,  $D^k$  satisfies  $\text{rank } D^k = m + 1$ ,  $\text{rank } \partial^{i+1} D^k = \text{rank } \partial^i D^k + m$  for  $i = 0, \dots, k$  and  $\partial^{k+1} D^k = T(P^k(R))$ .*

Schematically we have the following diagram;

$$\begin{array}{ccccccc}
 D^k & \subset & \partial D^k & \subset & \dots & \subset & \partial^{k-1} D^k & \subset & \partial^k D^k & \subset & \partial^{k+1} D^k = T(P^k(R)) \\
 \downarrow p_*^k & & & & & & \downarrow p_*^k & & \downarrow p_*^k & & \downarrow p_*^k \\
 D^{k-1} & \subset & \dots & \subset & \partial^{k-2} D^{k-1} & \subset & \partial^{k-1} D^{k-1} & \subset & \partial^k D^{k-1} = T(P^{k-1}(R)) \\
 & & \downarrow p_*^{k-1} & & \downarrow p_*^{k-1} & & \downarrow p_*^{k-1} & & \downarrow p_*^{k-1} & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & & & \vdots & & \vdots & & \vdots & & \\
 & & & & \downarrow p_*^2 & & \downarrow p_*^2 & & \downarrow p_*^2 & & \\
 & & & & D^1 & \subset & \partial D^1 & \subset & \partial^2 D^1 = T(P^1(R)) \\
 & & & & & & \downarrow p_*^1 & & \downarrow p_*^1 & & \\
 & & & & & & D & \subset & \partial D = T(R)
 \end{array}$$

where  $p^i : P^i(R) \rightarrow P^{i-1}(R)$  is the projection. Here we note

$$\partial^k D^k = (p_0^k)^{-1}(D),$$

where  $p_0^k : P^k(R) \rightarrow R$  is the projection.

#### 4. SPECIAL $m$ -FLAGS OF LENGTH $k$

An  $m$ -flag  $(R, D)$  ( $m \geq 2$ ) of length  $k$  is called a *special  $m$ -flag* if there exists a completely integrable subbundle  $F$  of  $\partial^{k-1} D$  of corank 1, which contains  $\text{Ch}(\partial^{k-1} D)$ , and  $\text{Ch}(\partial^i D)$  is a subbundle of  $\partial^{i-1} D$  of corank 1 for  $i = 1, \dots, k-1$ , such that  $\text{Ch}(D)$  is trivial, i.e., if the following diagram holds for  $(R, D)$ ;

$$\begin{array}{ccccccc}
 D & \subset & \partial D & \subset \dots \subset & \partial^{k-2} D & \subset \partial^{k-1} D & \subset \partial^k D = T(R) \\
 \cup & & \cup & & \cup & & \cup \\
 \{0\} = \text{Ch}(D) & \subset & \text{Ch}(\partial D) & \subset \dots \subset & \text{Ch}(\partial^{k-1} D) & \subset & F
 \end{array}$$

where  $\text{rank } \partial^i D = \text{rank } \partial^{i-1} D + m$  for  $i = 1, \dots, k$  and  $\text{rank } D = m + 1$ .

First, by repeated use of Rank 1 prolongations starting from a manifold  $M$  of dimension  $m + 1$ , we obtain by Proposition 3.1,

**Proposition 4.1.**  $(P^k(M), C^k)$  is a special  $m$ -flag of length  $k$ .

Conversely, by utilizing the following Realization Lemma, it is known that every special  $m$ -flag of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$ .

**Realization Lemma ([Y1;p.122])** *Let  $R$  and  $M$  be manifolds. Assume that the quadruple  $(R, D, p, M)$  satisfies the following conditions:*

- (i)  $p$  is a map of  $R$  into  $M$  of constant rank.
- (ii)  $D$  is a differential system on  $R$  such that  $F = \text{Ker } p_*$  is a subbundle of  $D$  of corank  $n$ .

*Then there exists a unique map  $\psi$  of  $R$  into  $J(M, n)$  satisfying  $p = \pi \cdot \psi$  and  $D = \psi_*^{-1}(C)$ . Furthermore, let  $u$  be any point of  $R$ . Then  $\psi$  is in fact defined by*

$$\psi(u) = p_*(D(u)) \text{ as a point of } \text{Gr}(T_x(M), n), \quad x = \pi(u),$$

*and satisfies  $\text{Ker}(\psi_*)_u = F(u) \cap \text{Ch}(D)(u)$ .*

**Theorem 4.2.** *A special  $m$ -flag  $(R, D)$  of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$ , where  $M$  is a manifold of dimension  $m + 1$ . Especially  $F$  is unique for  $(R, D)$ .*

**Remark 4.3.** After [MZ], P. Mormul first defined the notion of special  $m$ -flags of length  $k$  for  $m \geq 2$  in a slightly different form in [M2] (cf. Theorem 6.2), generalizing the works of [KR] or [PR]. The above theorem was first observed by him in Remark 3 [M2].

In view of Theorem 4.2, our task is to characterize the special  $m$ -flags among  $m$ -flags of length  $k$ , which will be accomplished in the following sections.

## 5. MAIN THEOREM ( $m \geq 3$ )

Let  $(R, D)$  be an  $m$ -flag of length 1, i.e.,  $R$  is a manifold of dimension  $2m + 1$  such that  $\text{rank } D = m + 1$  and  $\partial D = T(R)$ . By definition,  $(R, D)$  is a special  $m$ -flag ( $m \geq 2$ ) if there exists a completely integrable subbundle  $F$  of  $D$  of corank 1 and  $\text{Ch}(D)$  is trivial. Then, by Realization Lemma,  $(R, D)$  is locally isomorphic to  $(P^1(M), C^1) = (J(M, 1), C)$ , where  $M = R/F$  is (locally) the leaf space of the foliation  $F$  on  $R$ . In case  $m = 1$ , it is easy to see that a 1-flag of length 1 is a contact manifold of dimension 3. 2-flags of length 1 have peculiar aspects and were extensively studied in [C] (cf. §6). Now we start with the following characterization of special  $m$ -flags of length 1 for  $m \geq 3$ .

**Proposition 5.1.** *An  $m$ -flag  $(R, D)$  of length 1 for  $m \geq 3$  is a special  $m$ -flag if and only if  $D$  is of Cartan rank 1.*

Here, the *Cartan rank* of  $(R, D)$  is the smallest integer  $\rho$  such that there exist 1-forms  $\{\pi^1, \dots, \pi^\rho\}$ , which are independent modulo  $\{\eta^1, \dots, \eta^m\}$  and satisfy

$$d\alpha \wedge \pi^1 \wedge \dots \wedge \pi^\rho \equiv 0 \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \forall \alpha \in \mathcal{D}^\perp = \Gamma(D^\perp),$$

where  $D = \{\eta^1 = \dots = \eta^m = 0\}$ .



*Remark 5.2.* As a characterization of 1-jet spaces, Bryant's normal form theorem is well known ([B], [BCG3]). This theorem in 1 independent variable case says that an  $m$ -flag  $(R, D)$  of length 1 for  $m \geq 3$  is a special  $m$ -flag if and only if  $D$  is of Engel (half-) rank 1 and  $\text{Ch}(D)$  is trivial. Here the Engel rank of  $(R, D)$  is the smallest integer  $\rho$  such that

$$(d\alpha)^{\rho+1} \equiv 0 \pmod{\eta^1, \dots, \eta^m} \quad \text{for } \forall \alpha \in \mathcal{D}^\perp,$$

where  $D = \{\eta^1 = \dots = \eta^m = 0\}$ . Here we observe that we cannot replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m = 3$ , as the following example shows; Let  $(y^1, y^2, y^3, x^0, x^1, x^2, x^3)$  be a coordinate system of  $R$ . Let us take a coframe  $\{\eta^1, \eta^2, \eta^3, \theta^i, (i = 0, 1, 2, 3)\}$  as follows;

$$\eta^1 = dy^1 + x^2 dx^3, \quad \eta^2 = dy^2 + x^3 dx^1, \quad \eta^3 = dy^3 + x^1 dx^2, \quad \theta^i = dx^i.$$

Then, for  $D = \{\eta^1 = \eta^2 = \eta^3 = 0\}$ , we have

$$\begin{cases} d\eta^1 \equiv \theta^2 \wedge \theta^3 \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^2 \equiv \theta^3 \wedge \theta^1 \pmod{\eta^1, \eta^2, \eta^3}, \\ d\eta^3 \equiv \theta^1 \wedge \theta^2 \pmod{\eta^1, \eta^2, \eta^3}. \end{cases}$$

Thus  $(R, D)$  is a 3-flag of length 1 such that  $(R, D)$  is of Engel rank 1 and has non-trivial  $\text{Ch}(D)$ .

However we can replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m \geq 4$ , as the following Lemma implies.

**Lemma 5.3.** *Let  $V$  be a vector space over  $\mathbb{R}$ . Let  $\omega_1, \dots, \omega_r \in \wedge^2 V$  be 2-forms such that  $\{\omega_1, \dots, \omega_r\}$  are linearly independent and  $\omega_i \wedge \omega_j = 0$  for  $1 \leq i \leq j \leq r$ . Then*

(1) *In case  $r = 2$ . There exist vectors  $v_0, v_1, v_2 \in V$ , which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2.$$

(2) *In case  $r = 3$ . Either of the followings holds*

(i) *There exist vectors  $v_1, v_2, v_3 \in V$ , which are linearly independent, such that*

$$\omega_1 = v_2 \wedge v_3, \quad \omega_2 = v_3 \wedge v_1, \quad \omega_3 = \pm v_1 \wedge v_2.$$

(ii) *There exist vectors  $v_0, v_1, v_2, v_3 \in V$ , which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2, \quad \omega_3 = v_0 \wedge v_3.$$

(3) *In case  $r \geq 4$ . There exist vectors  $v_0, \dots, v_r \in V$ , which are linearly independent, such that*

$$\omega_1 = v_0 \wedge v_1, \quad \omega_2 = v_0 \wedge v_2, \quad \dots, \quad \omega_r = v_0 \wedge v_r.$$

In case  $m = 1$ , every Goursat flag of length  $k$  ( $k \geq 2$ ) is a special 1-flag, i.e., the Sandwich Lemma holds automatically ([MZ]). By contrast, we need some condition for an  $m$ -flag of length 2 ( $m \geq 2$ ) to be special as the following example shows.

*Example 5.4.* Let  $R$  be a manifold of dimension  $3m+1$  ( $m \geq 2$ ), and let  $(x^\alpha, y^\beta, z^\beta)$  ( $\alpha = 0, 1, \dots, m, \beta = 1, \dots, m$ ) be a coordinate system on  $R$ . For a fixed  $a \in \{0, 1, \dots, m-2\}$ , let us take a coframe  $\{\eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m, \theta^0, \dots, \theta^m\}$  as follows;

$$\begin{aligned} \theta^\alpha &= dx^\alpha & \eta^\gamma &= dz^\gamma + y^\gamma dx^0 - \frac{1}{2}(x^0)^2 dx^\gamma & (\gamma = 1, \dots, m-a-1) \\ \zeta^\beta &= dy^\beta + x^0 dx^\beta & \eta^\delta &= dz^\delta + y^{\delta-1} dx^{\delta-1} & (\delta = m-a, \dots, m) \end{aligned}$$

We consider  $D = \{\eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^m = 0\}$ . Then we have

$$\begin{aligned} \begin{cases} d\eta^\beta &\equiv 0 \\ d\zeta^\beta &\equiv \theta^0 \wedge \theta^\beta \end{cases} \pmod{\eta^1, \dots, \eta^m, \zeta^1, \dots, \zeta^m} & \text{ for } \beta = 1, \dots, m, \\ \begin{cases} d\eta^\gamma &\equiv \zeta^\gamma \wedge \theta^0 \\ d\eta^\delta &\equiv \zeta^{\delta-1} \wedge \theta^{\delta-1} \end{cases} \pmod{\eta^1, \dots, \eta^m} & \text{ for } \gamma = 1, \dots, m-a-1, \\ & \text{ for } \delta = m-a, \dots, m. \end{aligned}$$

Hence we get

$$\partial D = \{\eta^1 = \dots = \eta^m = 0\}, \quad \partial^2 D = T(R)$$

$$\text{Ch}(\partial D) = \{\eta^1 = \dots = \eta^m = \zeta^1 = \dots = \zeta^{m-1} = \theta^0 = \theta^{m-a-1} = \dots = \theta^{m-1} = 0\}$$

Thus,  $(R, D)$  is an  $m$ -flag of length 2, but  $\text{Ch}(\partial D)$  is not a subbundle of  $D$ . Moreover  $\text{rank Ch}(\partial D)$  is  $m-a$ .

In order to get good control over  $\text{Ch}(\partial D)$ , we prepare the following proposition, which gives us the Sandwich Lemma for  $m \geq 3$ .

**Proposition 5.5.** *Let  $(R, D)$  be a regular differential system such that  $\text{rank } \partial^2 D = \text{rank } \partial D + m$  and  $\text{rank } \partial D = \text{rank } D + m$ . Assume  $m \geq 3$  and the Cartan rank of  $\partial D$  is 1, then  $\text{Ch}(\partial D)$  is a subbundle of  $D$  of corank 1. Moreover the Cartan rank of  $D$  is 1*

In view of Lemma 5.3, we can replace the Cartan rank 1 condition by the Engel rank 1 condition when  $m \geq 4$  (cf. Remark 5.6).

*Remark 5.6.* We cannot replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m = 3$ , as the following example shows; Let  $(z^1, z^2, z^3, y^1, y^2, y^3, x^0, x^1, x^2, x^3)$  be a coordinate system of  $R$ . Let us take a coframe  $\{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3, \theta^0, \theta^1, \theta^2, \theta^3\}$  as follows;

$$\begin{aligned} \eta^1 &= dz^1 + y^1 dx^0, & \eta^2 &= dz^2 + y^2 dy^1, & \eta^3 &= dz^3 + x^0 dy^2, & \theta^0 &= dx^0, & \theta^1 &= dx^1, \\ \zeta^1 &= dy^1 - x^1 dx^0, & \zeta^2 &= dy^2 - x^2 dx^0, & \zeta^3 &= dy^3 - x^3 dx^0, & \theta^2 &= dx^2, & \theta^3 &= dx^3. \end{aligned}$$

We consider  $D = \{\eta^1 = \eta^2 = \eta^3 = \zeta^1 = \zeta^2 = \zeta^3 = 0\}$ . Then we have

$$\begin{aligned} \begin{cases} d\eta^\beta &\equiv 0 \\ d\zeta^\beta &\equiv \theta^0 \wedge \theta^\beta \end{cases} \pmod{\eta^1, \eta^2, \eta^3, \zeta^1, \zeta^2, \zeta^3} & \text{ for } \beta = 1, 2, 3, \\ \begin{cases} d\eta^1 &\equiv \zeta^1 \wedge \theta^0 \\ d\eta^2 &\equiv (\zeta^2 + x^2 \theta^0) \wedge (\zeta^1 + x^1 \theta^0) \\ d\eta^3 &\equiv \theta^0 \wedge \zeta^2 \end{cases} \pmod{\eta^1, \eta^2, \eta^3} & \begin{aligned} & \pmod{\eta^1, \eta^2, \eta^3}, \\ & \pmod{\eta^1, \eta^2, \eta^3}, \\ & \pmod{\eta^1, \eta^2, \eta^3}. \end{aligned} \end{aligned}$$

Hence we get

$$\begin{aligned}\partial D &= \{\eta^1 = \eta^2 = \eta^3 = 0\}, \quad \partial^2 D = T(R), \\ \text{Ch}(\partial D) &= \{\eta^1 = \eta^2 = \eta^3 = \zeta^1 = \zeta^2 = \theta^0 = 0\}.\end{aligned}$$

Thus,  $(R, D)$  is an 3-flag of length 2 such that the Engel rank of  $\partial D$  is 1, but  $\text{Ch}(\partial D)$  is not a subbundle of  $D$ .

However, by Lemma 5.3, we can replace the Cartan rank 1 condition in the above Proposition by the Engel rank 1 condition when  $m \geq 4$ .

By utilizing the above proposition repeatedly, we obtain

**Theorem 5.7.** *An  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 3$  is a special  $m$ -flag if and only if  $\partial^{k-1}D$  is of Cartan rank 1. Moreover, an  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 4$  is a special  $m$ -flag if and only if  $\partial^{k-1}D$  is of Engel rank 1.*

Hence, by Theorem 4.2, we obtain the Drapeau Theorem for  $m \geq 3$

**Corollary 5.8.** *Let  $M$  be a manifold of dimension  $m + 1$ . An  $m$ -flag  $(R, D)$  of length  $k$  for  $m \geq 3$  is locally isomorphic to  $(P^k(M), C^k)$  if and only if  $\partial^{k-1}D$  is of Cartan rank 1, and, moreover for  $m \geq 4$ , if and only if  $\partial^{k-1}D$  is of Engel rank 1.*

## 6. INTEGRABLE SUBBUNDLE OF CORANK 1

Let  $(R, D)$  be a 2-flag of length 1. Then it can be shown ([C]) that there exists a local coframe  $\{\eta^1, \eta^2, \theta^0, \theta^1, \theta^2\}$  such that  $D = \{\eta^1 = \eta^2 = 0\}$ ,

$$\begin{cases} d\eta^1 \equiv \theta^0 \wedge \theta^1 & (\text{mod } \eta^1, \eta^2), \\ d\eta^2 \equiv \theta^0 \wedge \theta^2 & (\text{mod } \eta^1, \eta^2) \end{cases}$$

Thus the Cartan rank of  $(R, D)$  is always 1 and we have the covariant system  $F = \{\eta^1 = \eta^2 = \theta^0 = 0\}$  of  $D$  of corank 1 (cf. [Y3]). As is well known,  $F$  is not necessarily completely integrable.

As for a 2-flag of length 2, we observe that, in Example 5.4, putting  $m = 2$ , we obtain the following structure equation for  $D = \{\eta^1 = \eta^2 = \zeta^1 = \zeta^2 = 0\}$ , where

$$\begin{aligned}\eta^1 &= dz^1 + y^1 dx^0 - \frac{1}{2}(x^0)^2 dx^1, \quad \eta^2 = dz^2 + y^1 dx^1 \\ \zeta^1 &= dy^1 + x^0 dx^1, \quad \zeta^2 = dy^2 + x^0 dx^2, \quad \theta^0 = dx^0, \quad \theta^1 = dx^1, \quad \theta^2 = dx^2,\end{aligned}$$

$$\begin{cases} d\eta^\beta \equiv 0 & (\text{mod } \eta^1, \eta^2, \zeta^1, \zeta^2) \text{ for } \beta = 1, 2, \\ d\zeta^\beta \equiv \theta^0 \wedge \theta^\beta & (\text{mod } \eta^1, \eta^2, \zeta^1, \zeta^2) \text{ for } \beta = 1, 2. \end{cases}$$

$$\begin{cases} d\eta^1 \equiv \zeta^1 \wedge \theta^0 & (\text{mod } \eta^1, \eta^2), \\ d\eta^2 \equiv \zeta^1 \wedge \theta^1 & (\text{mod } \eta^1, \eta^2). \end{cases}$$

Thus  $\partial D = \{\eta^1 = \eta^2 = 0\}$  and the Cartan rank of  $\partial D$  is 1, whereas  $\text{Ch}(\partial D)$  is not a subbundle of  $D$ . This shows that the statement of Proposition 5.5 is false for  $m = 2$ .

To cover the case  $m = 2$ , we strengthen the hypothesis of Proposition 5.5 as in the following.

**Proposition 6.1.** *Let  $(R, D)$  be a regular differential system such that  $\text{rank } \partial^2 D = \text{rank } \partial D + m$  and  $\text{rank } \partial D = \text{rank } D + m$ . Assume that there exists a completely integrable subbundle  $F$  of  $\partial D$  of corank 1, then  $\text{Ch}(\partial D)$  is a subbundle of  $D$  of corank 1.*

By utilizing the above proposition repeatedly, we obtain

**Theorem 6.2.** *An  $m$ -flag  $(R, D)$  of length  $k$  is a special  $m$ -flag if and only if there exists a completely integrable subbundle  $F$  of  $\partial^{k-1} D$  of corank 1. Moreover,  $F$  is unique for  $(R, D)$ .*

**Corollary 6.3.** *Let  $M$  be a manifold of dimension  $m + 1$ . An  $m$ -flag  $(R, D)$  of length  $k$  is locally isomorphic to  $(P^k(M), C^k)$  if and only if there exists a completely integrable subbundle  $F$  of  $\partial^{k-1} D$  of corank 1.*

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